Recall: A **basis** of vector space V is any subset $B \subseteq V$ such that ① B is lin. ind. ② B spans V.

"The only lin comb. giving 0_V is the zero-combination" *

"Every vector in V is a linear comb. of vectors from B" *

Prop: B is a basis of V iff every vector of V arises as a unique lin. comb of elts from B.

Recall: $\dim(V) \overset{*}{=}$ number of elements in a basis for V.

Ex: $\mathbb{R}^n$ has dimension n: $\mathcal{E}_n = \{e_1, e_2, \dots, n\}$.

Recall: $L : V \to W$ is **linear** when for all $u, v \in V$ and all $c \in \mathbb{R}$ we have $L(u + c \cdot v) \overset{*}{=} L(u) + c \cdot L(v)$.

NB: easiest condition to check...

The **rank** of L is $\dim(\mathrm{ran}(L))$.
The **nullity** of L is $\dim(\ker(L))$. *

→ **range** of L is $\mathrm{ran}(L) = \{L(v) : v \in V\}$
↳ i.e. set of outputs of function L.

→ **kernel** of L is $\ker(L) = \{v \in V : L(v) = 0_W\}$
↳ i.e. set of vectors mapping to 0_V under L.

Rank-Nullity Formula: $\dim(\mathrm{dom}(L)) = \mathrm{rank}(L) + \mathrm{nullity}(L)$.

Ex: $D = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$. Show D is dependent.

Method: $a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ solve!

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

No leading 1.

<u>Note:</u> leading 1's are in columns 1, 2.

So $\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$ is a <u>basis</u> of span(D).

Thus D is linearly dependent. $\square$

$$\operatorname{span}(D) = \left\{ a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

We can rewrite $\operatorname{RREF}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ in the system: $\begin{cases} a + c = 0 \\ b - c = 0 \end{cases} \rightsquigarrow \begin{cases} a = -c \\ b = c \end{cases}$

Point $-c\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\therefore$ when $c = 1$: $-\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0_v$

$v_3 = v_2 - v_1$

$$\therefore \operatorname{span}(D) = \{ av_1 + bv_2 + cv_3 : a, b, c \in \mathbb{R} \}$$
$$= \{ av_1 + bv_2 + c(v_2 - v_1) : a, b, c \in \mathbb{R} \}$$

$$= \{ \underbrace{(a-c)}\, v_1 + \underbrace{(b+c)}\, v_2 : a, b, c \in \mathbb{R} \}$$

$$= \{ \alpha v_1 + \beta v_2 : \alpha, \beta \in \mathbb{R} \}.$$

$$\operatorname{span}(D) = \operatorname{span}(D \setminus \{v_3\})$$

$$a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} + c\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\rightarrow \begin{array}{ccc} a & b & c \\ \downarrow & \downarrow & \downarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 1 & 1 & 0 & y \\ 0 & 1 & -1 & z \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 1 & -1 & -x+y \\ 0 & 1 & -1 & z \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 1 & -1 & -x+y \\ 0 & 0 & 0 & \boxed{x-y+z} \end{array}\right]$$

condition on the span :)

$\Rightarrow$ D is <u>not</u> spanning b/c for $x=y=0$, $z=1$ the system is not solvable... b/c $x-y+z = 1 \neq 0$.

i.e. $\begin{pmatrix}0\\0\\1\end{pmatrix} \notin \operatorname{span}(D)$.

So $\operatorname{span}(D) = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 : x-y+z=0 \right\}$

Ex: Is $S = \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\1&1\end{pmatrix} \right\}$ ~~lin. ind.~~ basis ?

Sol: $x\begin{pmatrix}1&0\\0&1\end{pmatrix} + y\begin{pmatrix}1&1\\0&0\end{pmatrix} + z\begin{pmatrix}0&1\\1&0\end{pmatrix} + w\begin{pmatrix}0&0\\1&1\end{pmatrix} = \begin{pmatrix}a&b\\c&d\end{pmatrix}$

iff $\begin{pmatrix} x+y & y+z \\ z+w & x+w \end{pmatrix} = \begin{pmatrix} 0\,a & 0\,b \\ 0\,c & 0\,d \end{pmatrix}$

Legend: pink replaces zeros...

iff $\begin{cases} x+y = 0\,a \\ \quad y+z = 0\,b \\ \qquad z+w = 0\,c \\ x \qquad\; +w = 0\,d \end{cases}$

so solve system: $\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0\,a \\ 0 & 1 & 1 & 0 & 0\,b \\ 0 & 0 & 1 & 1 & 0\,c \\ 1 & 0 & 0 & 1 & 0\,d \end{array}\right]$

~~show this has only 0-solution, so they're LI.~~ $\cancel{\boxed{\checkmark}}$

Ex: $L : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$

$$L(a+bx+cx^2+dx^3) = \begin{bmatrix} a+b & b+c \\ c+d & a+d \end{bmatrix}.$$

Compute $\ker(L)$ and $\operatorname{ran}(L)$ (give bases!).

Sol: First compute $\ker(L)$.

$a+bx+cx^2+dx^3 \in \ker(L)$

$\iff L(a+bx+cx^2+dx^3) = 0_W$

$\iff \begin{bmatrix} a+b & b+c \\ c+d & a+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\iff \begin{cases} a+b = 0 \\ b+c = 0 \\ c+d = 0 \\ a+d = 0 \end{cases} \rightsquigarrow$ reduce $\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right]$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \begin{cases} a + d = 0 \\ b - d = 0 \\ c + d = 0 \\ 0 = 0 \end{cases} \rightsquigarrow \begin{cases} a = -t \\ b = t \\ c = -t \\ d = t \end{cases}$$

So $\ker(L) = \{-t + tx - tx^2 + tx^3 : \underline{t \in \mathbb{R}}\}$

$= \{t(-1 + x - x^2 + x^3) : t \in \mathbb{R}\}$

$= \operatorname{span}\{-1 + x - x^2 + x^3\}.$

Hence $\{-1 + x - x^2 + x^3\}$ is a basis of $\ker(L)$.

<u>NB</u>: $\#\{-1 + x - x^2 + x^3\} = 1$, $\operatorname{nullity}(L) = \underline{1}$.

To compute a basis for range:

$\operatorname{ran}(L) \overset{*}{=} \{L(v) : v \in \operatorname{dom}(L)\}$

$\underset{*}{=} \{L(a + bx + cx^2 + dx^3) : a, b, c, d \in \mathbb{R}\}$

$$= \left\{ \begin{bmatrix} a+b & b+c \\ c+d & a+d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

$$= \operatorname{span} \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{show LI.}} \right\}$$

$\therefore \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis.

So $\operatorname{rank}(L) = 3$.

Alt-Rank Computation: $\dim(\operatorname{dom}(L)) = 4$

$\operatorname{nullity}(L) = 1$

$\therefore$ by Rank-nullity formula: $4 = 1 + \operatorname{rank}(L)$

$\therefore \operatorname{rank}(L) = 3.$ $\blacksquare$